Journal of Electrical Systems and Information Technology 2 (2015) 347–357
www.elsevier.com/locate/jesit

LQG/LTR based reference tracking for a modular servo

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Available online 13 December 2015

Abstract

The paper presents an application of the LQG/LTR controller for a motion system – a laboratory modular servosystem. Basic results on LQ regulator and Kalman filter designs are briefly surveyed and tuning of their combination – the LQG controller is shown. Based on comparison of frequency properties of the LQR and LQG controllers, the LTR methodology is applied to fine-tune the LQG in order to improve its robustness. LQG/LTR speed controller for the modular servosystem plant was designed based on noisy measurements of the angular displacement. Simulation and experimental results illustrate implementation of the LQG controller.

Keywords: LQG/LTR; Kalman filter; Motion systems; Reference tracking

1. Introduction

Since the seminal paper by Doyle and Stein (1981) was published in 1981, the LQG/LTR method has become classical, being included in standard textbooks and verified by a huge number of applications. The point of this approach is based on the fact that using the observer has no effect on the closed loop transfer function but has a harmful influence on the robustness properties. The LQG/LTR method aims at modifying the Kalman filter so that the harmful effects on stability margins are attenuated by making the open loop transfer function of the plant with observer asymptotically approximate the one which this would without the observer included.

The paper deals with a practical implementation of the LQG/LTR controller for a laboratory plant – a modular servosystem (MSS). The LQG controller design procedure is presented, subsequently the LTR methodology is applied for the LQG to improve closed-loop robustness against process and measurement noises. LQG/LTR speed controller based on noisy measurements of angular displacement was designed and verified by simulations and experiments.

The paper is organized into following sessions: Theoretical Background provides overview of basic results on LQ and LQG controller designs and corresponding closed-loop properties; principle of LTR methodology is outlined. The

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Peer review under responsibility of Electronics Research Institute (ERI).

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jesit.2015.11.002

Case Study section deals with the design of a LQG/LTR speed regulator of the laboratory plant (a modular servosystem, MSS).

2. Theoretical background

Results from the optimal control design for linear systems with quadratic performance index (LQ problem) form the basis of modern control theory. Many systems are linear to begin with, while many nonlinear systems may be considered as linear when being operated near equilibrium.

State-feedback LQ regulator (LQR) drives the state vector $x(t)$ of a dynamic system from arbitrary initial conditions to zero (operating point). Asymptotic stability and required performance are provided by a proper choice of weighting matrices in quadratic performance index. The necessary prerequisite for using LQR is availability of all plant states for feedback. If this is not the case, an observer has to be designed to provide state estimates to which the LQ controller is applied instead to real states. In practice, the most frequently used version is the stochastic observer known as Kalman filter; it provides state vector estimate from available plant output measurements that are often corrupted by measurement noise and process noise specified by their statistical properties.

One of the most important problems in control is making a system output to track a reference input signal; this is called tracking or servo design problem. For a setpoint tracking the regulator can be converted into a tracker by adding additional feedforward terms; if the reference is not constant feedforward terms generally contain also its derivatives. A powerful tracker design technique that automatically yields the pre-compensator required to guarantee proper tracking for a large class of command inputs is the command generator tracker (CGT) based on incorporating a model of the reference dynamics into the control system (Lewis, 1992; Kozáková, 2014).

2.1. LQR design and frequency domain properties

Consider the state-space model of a plant

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t)$, $u(t)$, $y(t)$ are state, control and output vectors, A, B, C are state, control and output matrices of compatible dimensions, and J is a quadratic performance index

$$J = \int_0^{\infty} (x^T(t)Q_c x(t) + u^T(t)R_c u(t))\,dt \tag{2}$$

where Q_c, is a symmetric positive semidefinite weighting matrix and R_c a symmetric positive definite weighting matrix.

Optimal state feedback control law has the well-known form

$$u(t) = -K_c x(t), \quad K_c = R_c^{-1} B^T P_c \tag{3}$$

where P_c is solution to the algebraic Riccati equation

$$A^T P_c + P_c A - P_c B R_c^{-1} B^T P_c + Q_c = 0 \tag{4}$$

Important frequency domain LQR properties are obtained by simple manipulations of the Riccati equation (4) using the notation $\Phi(s) = (sI - A)^{-1}$ and $G_c = K_c \Phi(s) B$ for the resolvent of A, and the plant transfer function, respectively.

The resulting inequality (Doyle and Stein, 1981; Camacho et al., 1997; Lewis, 1992)

$$||1 + G_c(j\omega)|| > 1 \tag{5}$$

guarantees that the plant transfer function does not enter the unit circle centered in $(-1,0j)$ which results in guaranteed phase margin 60° and gain margin $(<0.5, \infty)$ of the closed-loop under LQR.

2.2. LQG design and frequency domain properties

If it is not feasible to design a full-state optimal feedback controller (not all states are available for measurement) a state observer has to be designed; if moreover process and measurement noises affect the plant, the stochastic state observer – Kalman filter – has to be used. The LQR implemented with estimated state variables is called LQG controller.

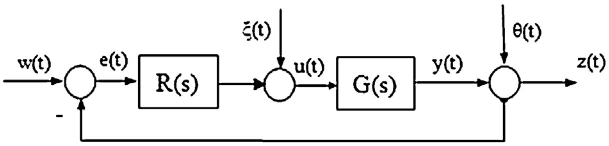

Fig. 1. Simple feedback control loop.

Due to separation principle, Kalman filter and LQR can be designed separately (Lee, 1995; Camacho et al., 1997; Lewis, 1992).

Consider the state-space model of a plant corrupted by the plant and the measurement noises $\xi(t)$ and $\theta(t)$, respectively.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \Gamma\xi(t) \\ y(t) &= Cx(t) + Du(t) + \theta(t) \end{aligned} \tag{6}$$

$\xi(t)$ and $\theta(t)$ are independent white Gaussian noises specified by their covariance matrices $Q_f \geq 0$ and $R_f > 0$, respectively.

Kalman filter design is a dual task to the LQR one. Denote the optimal state estimate $\hat{x}(t)$; then the estimation error is defined by

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{7}$$

The objective is to find the optimal estimate of the state vector $\hat{x}(t)$ to minimize the covariance of the state estimation error

$$E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T] = E[\tilde{x}(t)\tilde{x}(t)^T] \tag{8}$$

where the dynamics of the estimation error is given by the differential equation

$$\dot{\tilde{x}}(t) = (A - K_f C)\tilde{x}(t) + \Gamma\xi(t) - K_f\theta(t) \tag{9}$$

and of the optimal state estimate by

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K_f(y(t) - C\hat{x}(t)) \\ \dot{\hat{x}}(t) &= (A - K_f C)\hat{x}(t) + Bu(t) + K_f y(t) \end{aligned} \tag{10}$$

The Kalman filter gain matrix K_f is computed according to

$$\begin{aligned} K_f &= P_f C^T \Theta^{-1} \\ P_f &= P_f^T \geq 0 \end{aligned} \tag{11}$$

where P_f is solution to the filter Riccati equation

$$P_f A^T + AP_f - P_f C^T Q_f{}^{-1} C P_f + \Gamma R_f \Gamma^T = 0 \tag{12}$$

After having designed the Kalman filter, the LQG controller is obtained by implementing the LQ control law (3) using the estimated state

$$u(t) = -K_c\hat{x}(t) \tag{13}$$

The response of the closed-loop variable is equal to the one obtained using (3) for all states measurable.

2.3. Frequency domain closed-loop performance evaluation

The most commonly used frequency domain performance measures for SISO plants are the relative stability measures – gain and phase margins indicating at the same time closed-loop stability robustness. Using open-loop Bode plots it is possible to assess closed-loop performance as well as robustness against plant and measurement noises.

In the simple closed-loop structure in Fig. 1, plant noise $\xi(t)$ is acting at the plant input at low frequencies $\omega < \omega_\xi$ and the measurement noise $\theta(t)$ at high frequencies $\omega > \omega_\theta$. To guarantee proper reference tracking by the plant output, magnitude of the sensitivity has to be small for $\omega < \omega_\xi$ where the reference and the plant noise are "large"; in this way disturbance rejection is achieved (Lewis, 1992). To reject measurement noise, magnitude of the complementary sensitivity has to be small for $\omega > \omega_\theta$.

The above requirements can alternatively be formulated in terms of the magnitude of the loop transfer function $L = G(j\omega)R(j\omega)$; hence instead of calculating closed-loop sensitivities it is sufficient to use magnitude of the loop gain, i.e. open-loop Bode plots according to simple rules: at low frequencies $\omega < \omega_\xi$ the loop gain should be large, at high frequencies $\omega > \omega_\theta$ it should be small (Lewis, 1992). Performance in the crossover region is assessed from the roll-off rate.

2.4. LQG/LTR design

It is a well-known fact that the outstanding robustness properties of the LQR resulting from (5) considerably deteriorate if the LQR is implemented with a state observer (either the deterministic observer or the Kalman filter).

Loop Transfer Recovery (LTR) is a methodology to recover LQG robustness properties (in terms of open-loop gain and phase margins) to approach the LQR ones (Lee, 1995; Camacho et al., 1997; Lewis, 1992). Mathematically, this problem can be formulated using the open-loop transfer functions with LQR and LQG, respectively.

$$G_{\mathrm{LQ}}(s) = K_c(sI - A)^{-1}B \tag{14}$$

$$G_{\mathrm{LQG}}(s) = K_c(sI - A + BK_c + K_f C)^{-1} K_f C(sI - A)^{-1}B \tag{15}$$

Though (15) and (16) differ considerably, after suitable manipulations it is possible to find the way how they can approach each other.

It has been proved in (Lee, 1995; Camacho et al., 1997; Lewis, 1992) that by choosing new weighting matrices in (12) as follows

$$Q = Q_f + q^2 BVB^T \tag{16}$$

$$R = R_f \tag{17}$$

where V is a nonsingular matrix and $q \to \infty$ is a parameter, and solving the Riccati equation

$$P_f A^T + AP_f - P_f C^T Q^{-1} CP_f + \Gamma R \Gamma^T = 0 \tag{18}$$

it is possible to recover closed-loop robustness and performance under the LQG. Practical application of the above results is described in the next section.

3. Case study

3.1. Plant description and problem formulation

The Modular Servo System (MSS) consists of the INTECO digital servomechanism and open-architecture software environment for real-time control experiments (Modular Servo System, 2007). The measurement system is based on the RTDAC4/USB acquisition board. All functions of the board are accessed from the Modular Servo Toolbox.

MSS consists of the several modules arranged in the chain (Fig. 2): a DC motor with a generator, inertia load, encoder, magnetic brake and the gearbox with the output disk (in our experiments the backlash module was not applied). The servomechanism is connected to a computer where a control algorithm is based on measurements of the angular displacement and the angular velocity. For the above MSS, the LQG/LTR speed regulator is to be designed based on noisy measurements of angular displacement.

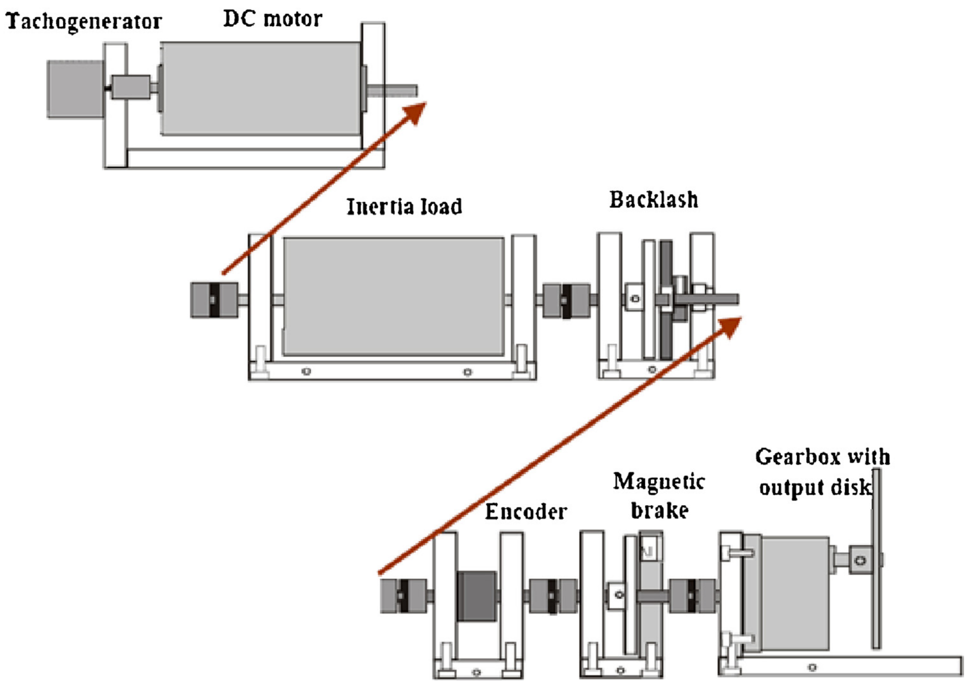

Fig. 2. Modular Servo System (MSS).

3.2. Plant identification

Based on I/O experiments, the plant was identified by a 3rd order state space model based on ARMAX polynomial model; φ is angular displacement, ω is the speed and x_3 is an auxiliary state.

$$\begin{bmatrix}\dot{\varphi}\\ \dot{\omega}\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -21.4139 & -12.9156\end{bmatrix}\begin{bmatrix}\varphi\\ \omega\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 3000\end{bmatrix} u$$

$$y = [100]\begin{bmatrix}\varphi\\ \omega\\ x_3\end{bmatrix}$$

3.3. LQG design

According to the separation principle, the LQG design was split into two independent steps: first, the LQR of speed was designed, using augmented state space model to include integrator

$$\begin{bmatrix}\ddot{\varphi}\\ \ddot{\omega}\\ \ddot{x}_3\\ \dot{e}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -0.001 & -21.414 & -12.916 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\varphi}\\ \dot{\omega}\\ \dot{x}_3\\ e\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 3000\\ 0\end{bmatrix}\dot{u}$$

Satisfactory closed-loop responses were obtained using weighting matrices $Q = \mathrm{diag}\{1,55,90,350\}$, $R = 51{,}000$ and the corresponding state feedback gain matrix $K_c = [0.0002, 0.0837\ 0.0386\ {-0.0827}]$.

In the second step, the Kalman filter was designed using noise covariances $Q = 0.02$, $R = 2 \times 10^{-5}$ calculated from measurements on the plant; Their fine-tuned values $Q_s = 1.6$, $R_s = 9.39 \times 10^{-4}$ were used to calculate the Kalman gain

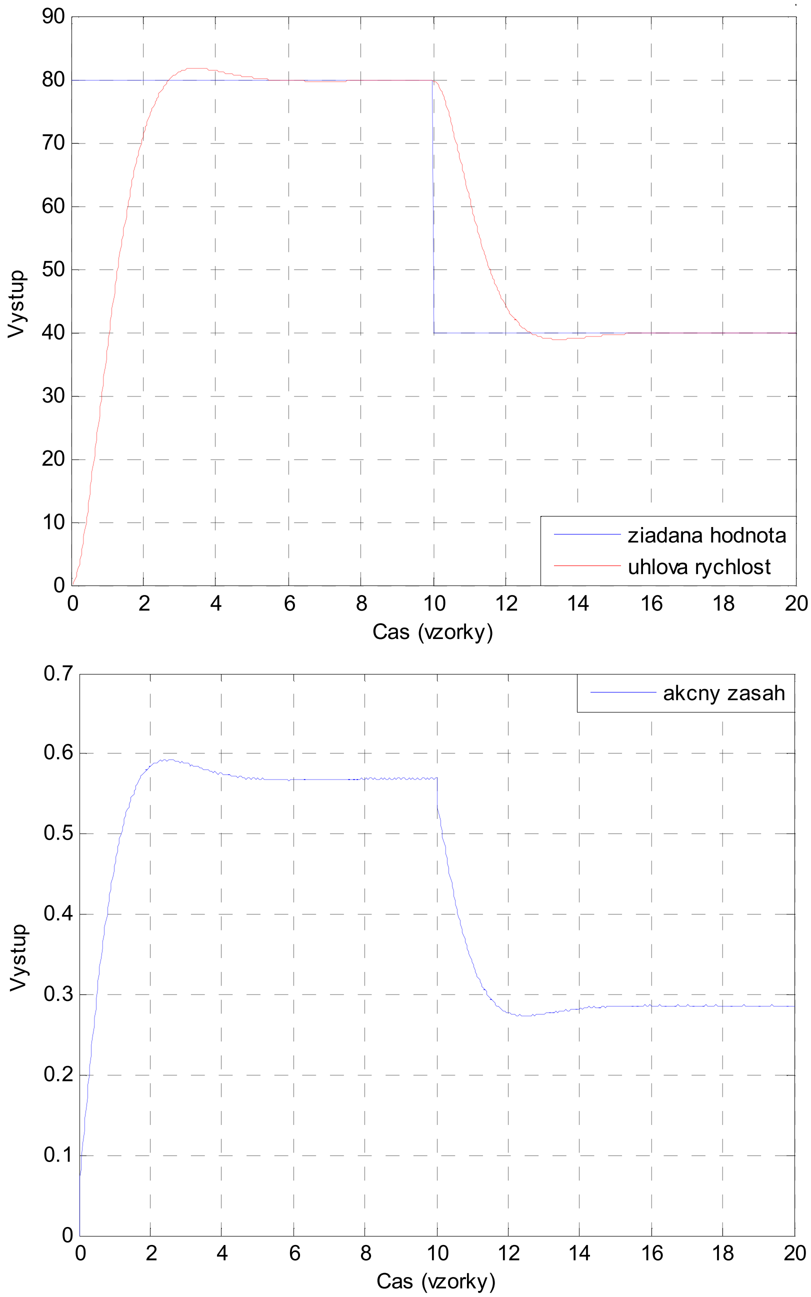

Fig. 3. Closed-loop responses under the LQG-PI controller: speed (upper plot), manipulated variable (lower plot) – simulation results.

$K_f = [42.281\ 41.665\ 15]T$. Simulation results are in Fig. 3. Settling time $ts = 4$ s, and maximum overshoot $\eta_{\max} = 4\%$ prove a satisfactory performance.

3.4. LQG/LTR design

Effect of applying the LTR procedure is evident from Tables 1 and 2 showing how the coefficient q affects Kalman filter gain and stability margins, respectively. This effect is apparent from Fig. 4. It is obvious that an excessive increase

Table 1
Kalman filter gain K_f depending on q.

q	Plant	10^2	10^4	10^6	10^8	10^{10}
K_f	$\begin{bmatrix}42.2809\\41.6653\\15.0001\end{bmatrix}$	$\begin{bmatrix}18.2883\\17.2319\\-2.2604\end{bmatrix}$	$\begin{bmatrix}174.207\\174.024\\141.168\end{bmatrix}$	$\begin{bmatrix}1733.05\\1733.03\\1697.96\end{bmatrix}$	$\begin{bmatrix}17321.5\\17321.5\\17286.2\end{bmatrix}$	$\begin{bmatrix}173206\\173206\\173171\end{bmatrix}$

Table 2
Gain margin G_m a phase margin P_m depending on q.

q	Open-loop under LQ	Open-loop under LQG	10^2	10^4	10^6	10^8	10^{10}
G_m	∞	8.9188	6.6753	11.9069	13.3562	13.5272	13.5445
P_m	109.822	61.918	61.1858	62.4689	62.6471	62.6656	62.667

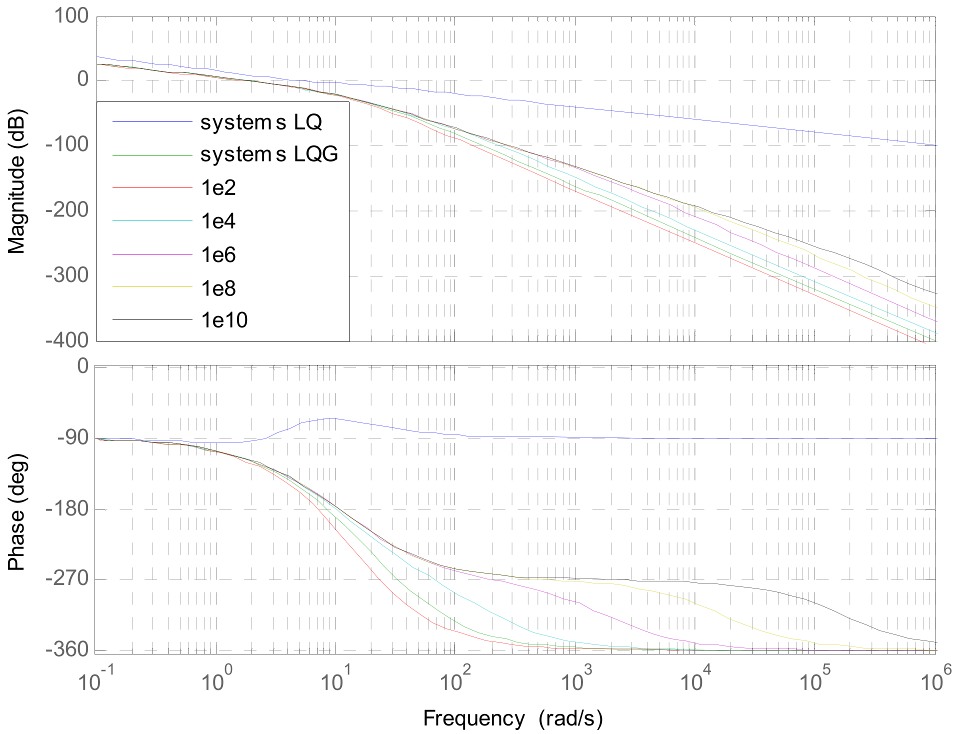

Fig. 4. LQG/LTR design: open – loop Bode plots for various values of q.

of q not only does not improve LQG robustness, but on the contrary, huge values of K_f bring about large and hence infeasible control actions and may even cause instability.

To avoid large control actions it is recommended to replace the reference steps by suitably constructed trapezoidal profile; this corresponds to ramp reference tracking. In such a case the control error has to be integrated twice. The resulting controller will be denoted as LQ-PII.

To be able to design LQ-PII, the original plant has to be augmented as follows:

$$A_{\text{aug}} = \begin{bmatrix} 0 & 1 & 0 & -0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.001 & -21.4139 & -12.9156 \end{bmatrix}$$

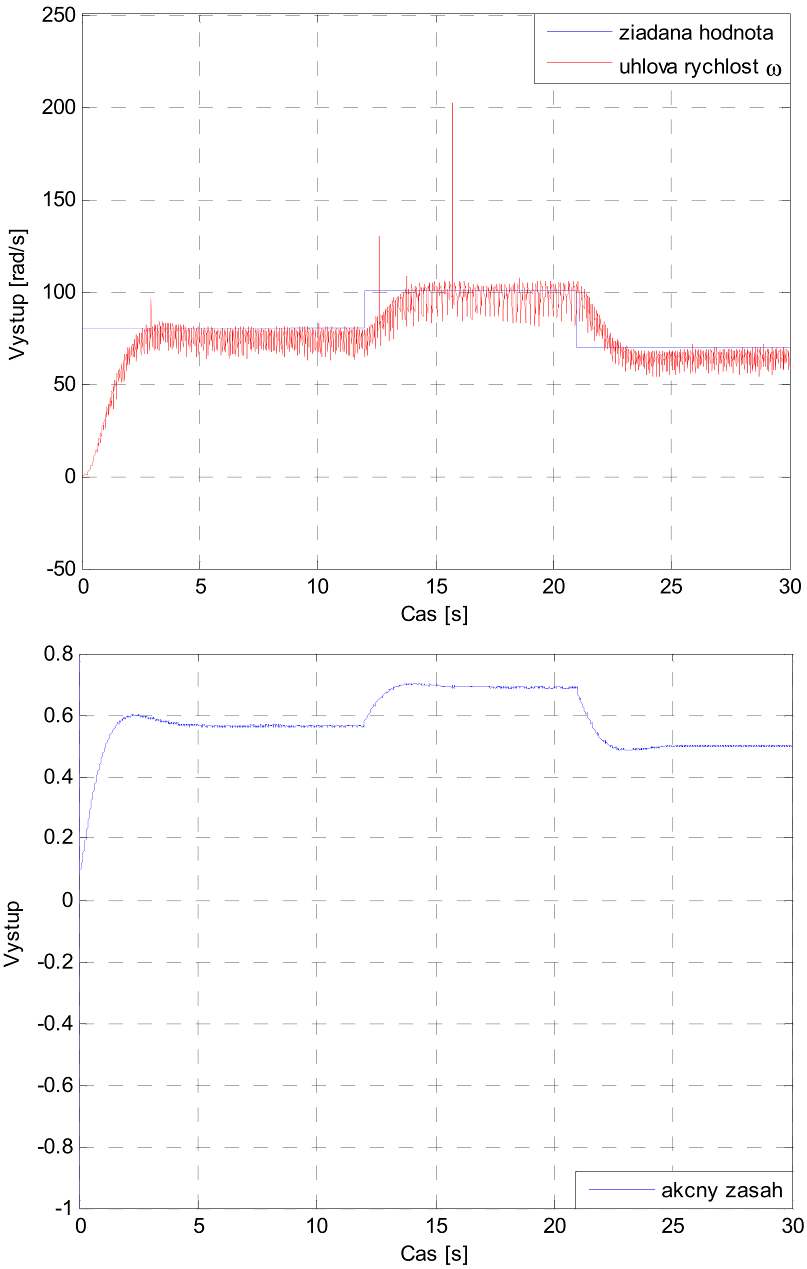

Fig. 5. Closed-loop responses under the LQG-PI controller (experimental results): responses of speed, and manipulated variable.

$$B_{\text{aug}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3000 \end{bmatrix} \quad Q_{\text{aug}} = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = 200$$

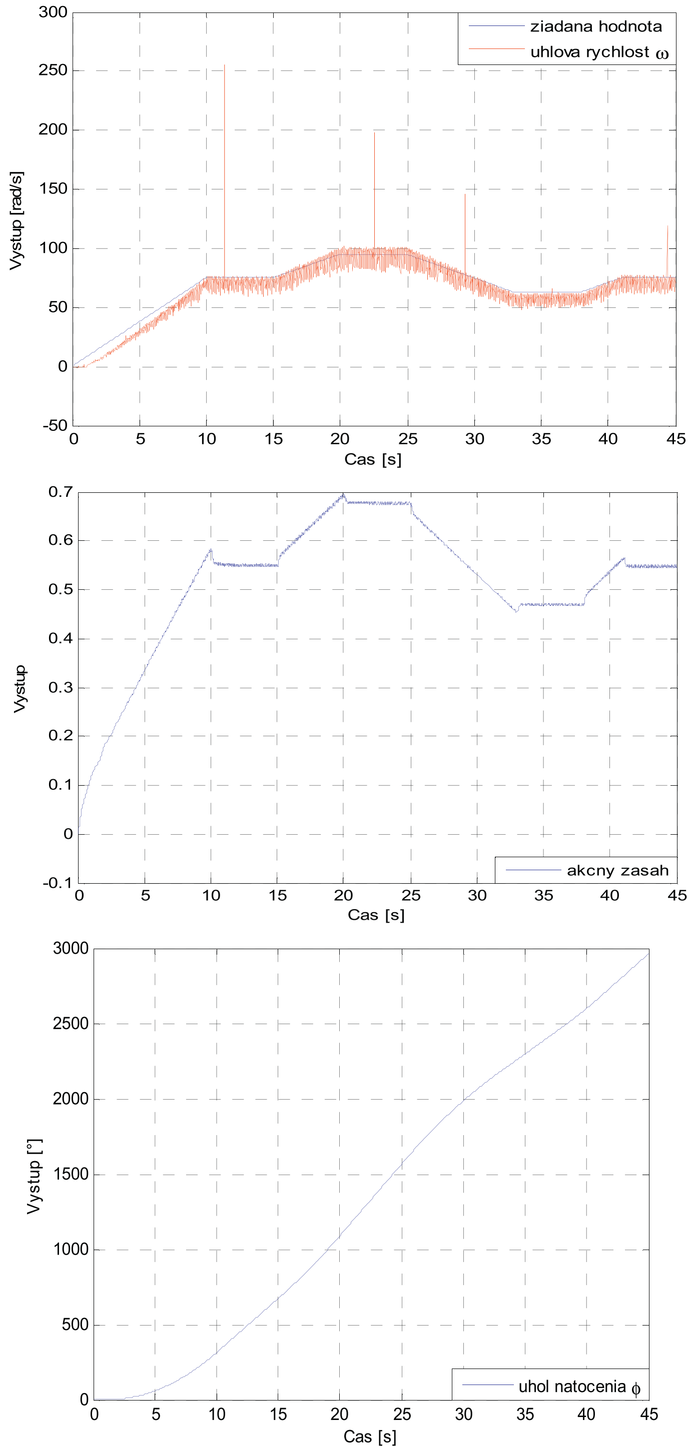

Fig. 6. Closed-loop responses under the LQG-PII controller (experimental results): responses of speed, angular displacement and manipulated variable.

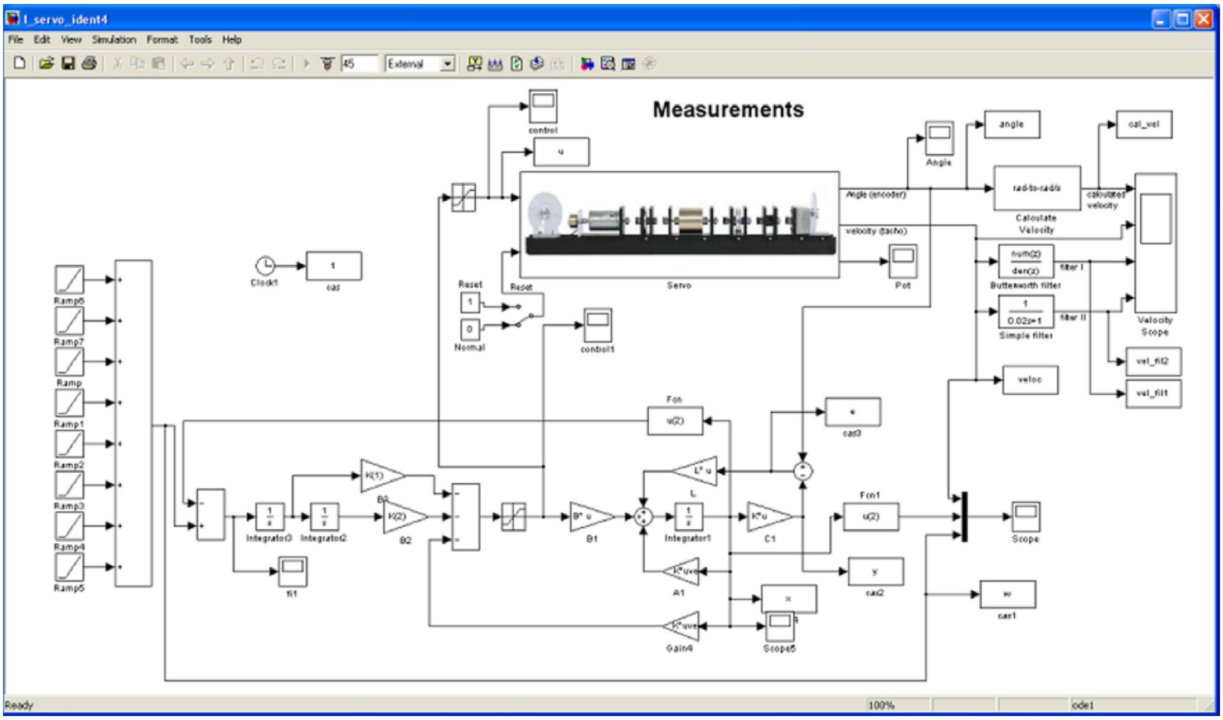

Fig. 7. Simulation scheme for experiments on the real plant (MSS) under the LQG-PII controller.

The resulting Kalman gain of the LQ-PII is as follows:

$$K_c = [-0.2000 - 0.3034\ 0\ 0.0605\ 0.0034]$$

Simulation scheme for experiments on the real plant (MSS) under the LQG-PII controller is in Fig. 7, closed-loop time responses are depicted in Fig. 6.

From the time response of the manipulated variable (control action) it is evident that its amplitudes are much smaller compare with Fig. 5, and hence better suited for practical implementation (Figs. 6 and 7).

4. Conclusion

The paper deals with LQG/LTR controller design for a physical laboratory plant. In general, the design consists of two (basically contradictory) steps:

1. Tuning LQG (LQR and Kalman filter independently) to achieve required performance of transition processes. Eigenvalues of $A - BK_c$ have to be slower than those of the Kalman filter matrix $A - K_fC$, achieved closed-loop performance should be comparable with the one under LQ state controller.
2. Improving robustness using the LTR methodology to make LQR and LQG frequency responses approach. The main limitation is that the resulting control has to be feasible to implement on the real plant.

Design of LQG/LTR speed controller to track step and ramp reference changes for the MSS based on noisy measurements of the angular displacement illustrates the theoretical outcomes.

Acknowledgement

The work on this paper was supported by the Slovak Research and Development Agency, project No. APVV-0772-12, and the Cultural and Educational Grant Agency of the Ministry of Education, Research and Sport of the Slovak Republic, grant No. 030STU-4/2015.

References

Camacho, E.F., Berenguel, M., Rubio, F.R., 1997. Advanced Control of Solar Plants. Springer, Sevilla.
Doyle, J.C., Stein, G., 1981. Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis. IEEE Trans AC-26, pp. 4–16.
Kozáková, A., 2014. Discrete-time asymptotic reference tracking for an industrial robot. In: In 23rd Int. Conference on Robotics in Alpe-Adria-Danube Region (RAADÍ14), Smolenice, Slovakia.
Lee, Y.J., 1995. The LQG/LTR dynamic digital control system design for the nuclear steam generator water level. J. Korean Nucl. Soc. 27.
Lewis, F.L., 1992. Applied Optimal Control & Estimation. Prentice Hall.
Modular Servo System, 2007. A User Manual. INTECO.